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Systems of Tautochrones in a General Field of Force.[*]

BY HARRY WILFRED REDDICK.

Introduction.

This paper deals with the system of curves each of which possesses the following property: A particle, starting from rest and moving along the curve, under the action of a given force, will reach a certain point, the center of tautochronous motion, in the same time from whatever point on the curve it starts. Friction is neglected and it is assumed that the acting force depends only on the position of the particle. The first part of the paper deals with plane systems. It is assumed that within the region considered the rectangular components of the force, $\phi(x, y)$, $\psi(x, y)$, are uniform functions which possess first and second derivatives and do not vanish identically.

In 1673 Huyghens found that when the acting force is gravity the tautochrones are cycloids with horizontal bases and concavity upwards. Since that time many extensions of the problem have been made and the tautochrones due to more complicated laws of force and resistance have been found. This paper, however, does not deal with the problem of finding the tautochrones due to a general positional force, but investigates the form of the differential equation and the derivable geometric properties of the system. This differential equation, which is of the third order and represents the total triply infinite system of tautochrones in the plane, is found in Article 1 from the well-known physical property of tautochrones, namely, that the tangential component of the acting force along the curve is proportional to the arcual distance of the moving particle from the center of tautochronous motion. The factor of proportionality must be negative for actual tautochrones, but, as it is eliminated in finding the differential equation of the system, the system includes virtual as well as actual tautochrones.

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[*] Presented, in part, before the American Mathematical Society, September 14, 1909, under the title, "Geometric Properties of a System of Tautochrones."

For this reason we choose to include virtual tautochrones in the system under consideration.

The geometric properties of the system are next investigated. At any point (x, y) of the plane in a given direction y' there is a singly infinite system of tautochrones. If the osculating parabola of each of these curves is constructed at the given point, the locus of their foci will be a bicircular quartic having a flecnode at the given point, the stationary tangent at this point having the given direction y'. The angle θ between the two tangents to the quartic at the given point is related in a definite way to the angle θ' which the acting force makes with the stationary tangent, namely $\tan\theta = 3\tan\theta'$.

The inverse of the quartic with respect to the given point (x, y) is a hyperbola, one of whose asymptotes is the stationary tangent to the quartic at the point (x, y); the other asymptote is parallel to the second tangent to the quartic at the point (x, y). The locus of the center of this hyperbola as the given direction y' varies is a circular cubic having the given point (x, y) as double point; the real asymptote of the cubic is parallel to the direction of the force acting at the given point. A characteristic property of conservative forces is that the tangents to this cubic at the given point are perpendicular to each other.

The next property pertains to hyperosculating circles. If the osculating circle of each tautochrone through a given point (x, y) is constructed at this point, it is found that the locus of the centers of those circles which hyperosculate their respective tautochrones is a cubic through the given point. The tangent to this cubic at the given point is the line along which the force acts at the point.

In the next section the question of the sufficiency of the preceding geometric properties for characterizing a system of tautochrones is considered. These properties characterize the system of tautochrones through a given point, up to differential elements of the third order. Additional relations are found which make the characterization of the total triply infinite system of tautochrones complete.

In Articles 16–18, systems of curves of different types, depending on their possession of certain of the preceding geometric properties, are considered with regard to the effect of point transformations on them. It is found that the only point transformations which transform the system possessing all the geometric properties under consideration into a system of the same type are the similitude transformations. Articles 19–20 deal with the system of tautochrones corresponding to a given value of the time constant.

In the last part of the paper, space tautochrones are considered and the differential equation of the totality of tautochrones in three-dimensional space is found. The class of forces for which every straight line is a tautochrone is determined. It is found that a known relation between such a class of forces in the plane and the class of collineations does not hold in space.[*] Finally, it is shown that the only conservative forces for which every straight line is a tautochrone are those for which the potential is a quadratic function of x, y and z.

The Differential Equation of the Tautochrones.

1. Suppose that at each point of the plane a force is acting whose rectangular components are functions only of the coordinates of the point, namely $\phi(x, y)$, $\psi(x, y)$. Such a force is called a positional force. We shall consider the system of tautochrones in a plane, due to a positional force; that is, the system of curves each of which possesses the following property: A particle which, starting from rest, moves along the curve under the action of a given positional force, will reach a certain point, the center of tautochronous motion, in the same time from whatever point on the curve it starts.

For such a curve we have the well-known property,

$$T = k(s - s_0), \tag{1}$$

where T is the tangential component of the force, $s - s_0$ is the arc measured from the center of tautochronous motion to the position of the moving particle, and k is the time constant whose value in terms of τ, the time required for a particle to move from any point of the curve to the center, is

$$k = -\frac{\pi^2}{4\tau^2}.$$

By differentiating equation (1) twice with respect to s, we obtain the intrinsic equation of the system of all tautochrones in the plane,[†] namely

$$Nr_s = T_1 r^2 + (T_2 + \bar{N})r - T, \tag{2}$$

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[*] Kasner: "Tautochrones and Brachistochrones," *Bulletin of the American Mathematical Society*, Vol. XV (1909), p. 480.

[†] Kasner: "Tautochrones," *loc. cit.*, p. 477.

where r is the radius of curvature, T and N are respectively the tangential and normal components of the force, and

$$\left.\begin{aligned} \bar{N} &= \frac{\psi_x + (\psi_y - \phi_x)\,y' - \phi_y\,y'^2}{1 + y'^2}, \\ T_1 &= \frac{\phi_{xx} + (2\,\phi_{xy} + \psi_{ax})\,y' + (\phi_{yy} + 2\,\psi_{xy})\,y'^2 + \psi_{yy}\,y'^3}{(1 + y'^2)^{\frac{3}{2}}}, \\ T_2 &= \frac{\phi_y + \psi_x + 2\,(\psi_y - \phi_x)\,y' - (\phi_y + \psi_x)\,y'^2}{1 + y'^2}, \end{aligned}\right\} \tag{3}$$

subscripts denoting partial differentiation and primes denoting differentiation with respect to x. If we substitute in equation (2) the values of $\bar{N}$, T_1 and T_2 given in equations (3), and the values of N, T, r and r_s given by

$$\left.\begin{aligned} r &= \frac{(1 + y'^2)^{\frac{3}{2}}}{y''}, \quad r_s = \frac{3\,y'\,y''^2 - (1 + y'^2)\,y'''}{y''^2}, \\ N &= \frac{\psi - y'\,\phi}{\sqrt{1 + y'^2}}, \quad T = \frac{\phi + y'\,\psi}{\sqrt{1 + y'^2}}, \end{aligned}\right\} \tag{4}$$

we obtain, after reduction, the Cartesian equation of the system of all tautochrones in the plane,

$$\left.\begin{aligned} y''' = {} & \frac{[\phi_{xx} + (2\,\phi_{xy} + \psi_{xx})\,y' + (\phi_{yy} + 2\,\psi_{xy})\,y'^2 + \psi_{yy}\,y'^3]\,(1 + y'^2)}{\phi\,y' - \psi} \\ & + \frac{\phi_y + 2\,\psi_x + 3\,(\psi_y - \phi_x)\,y' - (2\,\phi_y + \psi_x)\,y'^2}{\phi\,y' - \psi}\,y'' \\ & + \frac{3\,\phi\,y'^2 - 4\,\psi\,y' - \phi}{(\phi\,y' - \psi)\,(1 + y'^2)}\,y''^2. \end{aligned}\right\} \tag{5}$$

This is a differential equation of the third order and represents a triply infinite system of curves.

2. *Special case: Gravity.* When the acting force is gravity,

$$\phi = 0, \quad \psi = g.$$

Then equation (5) reduces to

$$y''' = \frac{4\,y'\,y''^2}{1 + y'^2}. \tag{6}$$

Equation (6) represents the totality of tautochrones in the plane due to gravity. It is the differential equation of the system of cycloids with horizontal bases, and may be obtained by taking the Cartesian equation of such a system, namely,

$$x + c = a \cos^{-1}\left(1 \mp \frac{y+d}{a}\right) + \sqrt{\pm 2a(y+d) - (y+d)^2},$$

and differentiating three times, eliminating c, a and d. The actual tautochrones of this system are the cycloids concave upwards corresponding to the upper signs in the above equation. The system of straight lines in the plane also belongs to the totality of tautochrones represented by equation (6). A straight line may be regarded as a tautochrone whose center is at infinity.

3. We shall now seek geometric properties of the system (5) and determine the necessary and sufficient conditions that a triply infinite system of curves shall be a system of tautochrones.

Geometric Properties of the System.

4. *The focal curve.* Consider first the system of tautochrones through a given point (x, y) of the plane in a given direction y'. This is a system of ∞^1 curves, each corresponding to a different value of the time constant Suppose that at the given point (x, y) the osculating parabola of each of these curves is constructed; that is, the parabola having four-point contact with the curve. Then with each of the ∞^1 curves of the system may be associated a point, namely, the focus of its osculating parabola constructed at the given point. The locus of the foci of these osculating parabolas is called the *focal curve.*[*] We now determine the focal curve of the above system of tautochrones.

The coordinates α, β of the focus of an osculating parabola, referred to the point (x, y) as origin, are given by the formulas

$$\alpha = \frac{-3y''}{2} \, \frac{y'''(y'^2-1) + 2y'(3y''^2 - y'y''')}{y'''^2 + (3y''^2 - y'y''')^2}, \tag{7}$$

$$\beta = \frac{-3y''}{2} \, \frac{(3y''^2 - y'y''')(y'^2-1) - 2y'y'''}{y'''^2 + (3y''^2 - y'y''')^2}. \tag{8}$$

We now eliminate y'' and y''' between these equations (7) and (8) and the equation of the tautochrones (5). From equations (7) and (8),

$$\frac{\alpha}{\beta} = \frac{y'''(y'^2-1) + 2y'(3y''^2 - y'y''')}{(3y''^2 - y'y''')(y'^2-1) - 2y'y'''}. \tag{9}$$

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[*] Kasner: "The Trajectories of Dynamics," *Transactions of the American Mathematical Society*, Vol. VII (1906), p. 405.

The elimination of y''' between equations (8) and (9) gives

$$y'' = -\frac{(\alpha y' - \beta)(1 + y'^2)}{2(\alpha^2 + \beta^2)}. \tag{10}$$

Substituting this value of y'' in (5), and the resulting value of y''' in (9), we obtain, after reduction,

$$\left.\begin{aligned} &4\left[\phi_{xx} + (2\phi_{xy} + \psi_{xx})y' + (\phi_{yy} + 2\psi_{xy})y'^2 + \psi_{yy}y'^3\right](\alpha^2 + \beta^2)^2 \\ &\quad - 2\left[\phi_y + 2\psi_x + 3(\psi_y - \phi_x)y' - (2\phi_y + \psi_x)y'^2\right](\alpha y' - \beta)(\alpha^2 + \beta^2) \\ &\quad - (\alpha y' - \beta)\left[(\psi y'^2 - 2\phi y' + 3\psi)\alpha - (3\phi y'^2 - 2\psi y' + \phi)\beta\right] = 0. \end{aligned}\right\} \tag{11}$$

This is the general equation of the focal curve, α and β being the coordinates of a point of the curve referred to the given point (x, y) as origin. The coefficients in equation (11) are determined by the coordinates x, y of the given point and the given direction y', since, for a positional force, ϕ and ψ are functions only of x and y.

An inspection of equation (11) shows that it represents a bicircular quartic having a node at the origin. The equations of the two tangents at the origin, t_1, t_2, say, are

$$t_1: \quad \alpha y' - \beta = 0, \tag{12}$$

$$t_2: \quad (\psi y'^2 - 2\phi y' + 3\psi)\alpha - (3\phi y'^2 - 2\psi y' + \phi)\beta = 0. \tag{13}$$

The tangent t_1 has contact of the third order with the quartic, cutting it at the origin in four coincident points, three on one branch and one on the other; it is, then, a stationary tangent. Its slope $\beta/\alpha = y'$. The other tangent t_2 has contact of the second order with the quartic, cutting it at the origin in three coincident points, two on one branch and one on the other. The origin is, then, a flecnode of the quartic, since one of the tangents is stationary.

THEOREM I. *At any point (x, y) of the plane in a given direction y', due to a positional force, there is a singly infinite system of tautochrones, each of which corresponds to a different value of the time constant. The focal curve of this system is a bicircular quartic having a flecnode at the origin, the stationary tangent having the given direction y'.*

The slope of the tangent t_2, from equation (13), is

$$\frac{\beta}{\alpha} = \frac{\psi y'^2 - 2\phi y' + 3\psi}{3\phi y'^2 - 2\psi y' + \phi}.$$

Taking the x axis in the direction y' of the tangent t_1, we have

$$\text{Slope of } t_2 = \frac{3\psi}{\phi};$$

but ψ/ϕ is the tangent of the angle which the direction of the acting force makes with the tangent t_1. Calling this angle θ' and the angle between t_1 and t_2, θ, we have

$$\tan\theta = 3\tan\theta'. \tag{14}$$

THEOREM II. *The second tangent t_2 to the quartic at the origin is such that the tangent of the angle between t_1 and t_2 is three times the tangent of the angle between t_1 and the direction of the acting force.*

5. *The focal curve for special forces.* When the acting force is gravity, then by substituting $\phi = 0$, $\psi = g$, in equation (11), the focal curve reduces to

$$(\alpha y' - \beta)\left[(y'^2 + 3)\alpha + 2y'\beta\right] = 0.$$

This represents two straight lines through the origin, and if θ is the angle between them, we find $\tan\theta = -3/y'$. When $y' = 0$, $\theta = 90°$ and the lines are the x and y axes. For actual tautochrones the focal curve consists only of a part of this locus, the upper half of the y axis.

When the coefficient of the fourth degree terms in equation (11) vanishes, the focal curve degenerates into the line t_1 and a circle through the origin tangent to the line t_2. This happens only when the force is that defined by the set of partial differential equations:

$$\left.\begin{array}{ll} \phi_{xx} = 0, & 2\phi_{xy} + \psi_{xx} = 0, \\ \phi_{yy} + 2\psi_{xy} = 0, & \psi_{yy} = 0. \end{array}\right\} \tag{15}$$

This class of forces is discussed by Kasner in the first paper cited; it is the class of forces for which every straight line in the plane is a tautochrone.

6. *The inverse curve.* We now take the inverse of the focal quartic with respect to the origin. Applying the formulas of inversion,

$$\alpha = \frac{\xi}{\xi^2 + \eta^2}, \quad \beta = \frac{\eta}{\xi^2 + \eta^2}, \quad \alpha^2 + \beta^2 = \frac{1}{\xi^2 + \eta^2}, \tag{16}$$

to equation (11), we obtain

$$\left.\begin{array}{c} [(3\psi - 2\phi y' + \psi y'^2)y']\xi^2 - [(3\psi - 2\phi y' + \psi y'^2) + (\phi - 2\psi y' + 3\phi y'^2)y']\xi\eta \\ + [\phi - 2\psi y' + 3\phi y'^2]\eta^2 + 2[\phi_y + 2\psi_x + 3(\psi_y - \phi_x)y' - (2\phi_y + \psi_x)y'^2](\xi y' - \eta) \\ - 4\,[\phi_{xx} + (2\phi_{xy} + \psi_{xx})y' + (\phi_{yy} + 2\psi_{xy})y'^2 + \psi_{yy}y'^3) = 0, \end{array}\right\} \tag{17}$$

which is the equation of the inverse curve. Here, ξ, η are the coordinates of a point of the inverse curve referred to the given point (x, y) as origin, and the coefficients are determined when x, y and y' are given. The curve is a conic and its discriminant

$$[(3\psi - 2\phi y' + \psi y'^2) - (\phi - 2\psi y' + 3\phi y'^2)y']^2 > 0;$$

hence the conic is a hyperbola.

We find the equations of the asymptotes of the hyperbola to be

$$\eta = y'\xi, \tag{18}$$

$$\eta = \frac{3\psi - 2\phi y' + \psi y'^2}{\phi - 2\psi y' + 3\phi y'^2}\xi + \frac{2[\phi_y + 2\psi_x + 3(\psi_y - \phi_x)y' - (2\phi_y + \psi_x)y'^2)}{\phi - 2\psi y' + 3\phi y'^2}. \tag{19}$$

Hence one asymptote passes through the origin in the given direction y'; it is the tangent t_1 to the quartic [equation (12)]. The other asymptote has the same slope as the tangent t_2 to the quartic [equation (13)].

The inverse of the focal quartic is a hyperbola. One of its asymptotes is the tangent t_1 to the quartic at the origin; the other asymptote is parallel to the other tangent t_2 to the quartic at the origin.

7. *The locus of the center of the inverse curve.* So far we have considered only the system of ∞^1 tautochrones that pass through a given point in a given direction. For such a system we have found that Theorems I and II hold. Now let the tangent t_1, in the given direction y', rotate about the given point (x, y). For each position of t_1 we have a system of ∞^1 tautochrones, such as that of Article 4, passing through the given point and tangent to t_1. We thus obtain a system of ∞^2 curves—all the tautochrones in the plane passing through a given point. For each position of t_1 there is a focal quartic of the system of tautochrones tangent to t_1 at the given point. The curve inverse to this quartic is a hyperbola, the coordinates of whose center, referred to the point (x, y) as origin, we will denote by X and Y. Hence, with each direction y' of t_1 may be associated a point (X, Y) of the plane. We now determine the locus of the point (X, Y) as y' varies.

Since the center of the hyperbola is on the line t_1 whose slope is y', we have

$$Y = y'X. \tag{20}$$

Applying the formula,

$$X = \frac{bg - hf}{h^2 - ab},$$

for the abscissa of the center of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

to the conic (17), we get, after reduction,

$$X = -\frac{2\,[\phi_y + 2\psi_x + 3(\psi_y - \phi_x)\,y' - (2\phi_y + \psi_x)\,y'^2]\,y'}{(3\psi - 2\phi y' + \psi y'^2) - (\phi - 2\psi y' + 3\phi y'^2)\,y'}. \tag{21}$$

Eliminating y' between equations (20) and (21), we obtain

$$3(\psi X - \phi Y)(X^2 + Y^2) + 2[(\phi_y + 2\psi_x)X^2 + 3(\psi_y - \phi_x)XY - (2\phi_y + \psi_x)Y^2] = 0. \tag{22}$$

This represents the locus of the centers of the hyperbolas inverse to the focal curves, X, Y being the coordinates of a point on the locus referred to the given point (x, y) as origin.

An inspection of equation (22) shows that it represents a circular cubic having a double point at the origin. Working out the equation of the real asymptote of the cubic, we obtain

$$Y = \frac{\psi}{\phi}X + \frac{2\,[(\phi_y + 2\psi_x)\phi^2 + 3(\psi_y - \phi_x)\phi\psi - (2\phi_y + \psi_x)\psi^2]}{3\phi(\phi^2 + \psi^2)}. \tag{23}$$

The slope of this line is the same as the slope of the line of force at the given point. The asymptote is therefore parallel to the direction in which the force is acting at the given point.

THEOREM III. *Through a given point (x, y) of the plane, in a given direction y', passes a singly infinite system of tautochrones, the inverse of the focal curve of the system, with respect to the given point (x, y), being a hyperbola. The locus of the center of this hyperbola as y' varies is a circular cubic having the given point as double point. The real asymptote of the cubic is parallel to the direction of the force acting at the given point.*

8. *The tangents to the cubic at the origin.* From equation (22), the tangents to the cubic at the origin are given by

$$Y = \frac{3(\psi_y - \phi_x) \pm \sqrt{9(\psi_y - \phi_x)^2 + 4(2\phi_y + \psi_x)(\phi_y + 2\psi_x)}}{2(2\phi_y + \psi_x)}X. \tag{24}$$

These tangents will be real and distinct, coincident, or imaginary; that is, the origin will be a node, cusp, or conjugate point of the cubic according as

$$D = 9(\psi_y - \phi_x)^2 + 4(2\phi_y + \psi_x)(\phi_y + 2\psi_x) >, =, < 0. \tag{25}$$

If γ represents the angle between these tangents, we have

$$\tan\gamma = \frac{\sqrt{9(\psi_y - \phi_x)^2 + 4(2\phi_y + \psi_x)(\phi_y + 2\psi_x)}}{\phi_y - \psi_x}. \tag{26}$$

Special cases. For a conservative force, $\phi_y = \psi_x$. In this case (25) gives

$$D = 9(\psi_y - \phi_x)^2 + 36\phi_y^2 > 0.$$

The origin is a node, and from (26), $\gamma = 90°$. Conversely, when $\gamma = 90°$, $\phi_y = \psi_x$. This property is therefore characteristic of conservative forces.

A system of tautochrones will be a system due to a conservative force when and only when the tangents at the origin to the cubic of Theorem III are perpendicular to each other.

For a so-called analytic force,[*] $\phi_x = \psi_y$, $\phi_y = -\psi_x$. In this case (25) gives $D = -4\psi_x^2 < 0$. The origin is a conjugate point, and from (26), $\tan\gamma = i$. Conversely, if $\tan\gamma = i$, $\phi_x = \psi_y$ and $\phi_y = -\psi_x$; hence the property is characteristic of analytic forces.

9. *Hyperosculating circles.* Suppose that the circle of curvature of each tautochrone of the doubly infinite system through the point (x, y) is constructed at that point. Out of the ∞^2 circles thus constructed we now determine which ones hyperosculate their respective tautochrones. For hyperosculation the contact must be of the third order; hence the value of y''' in the equation of the circle must be equal to the value of y''' in the equation of the tautochrone.

The differential equation of all circles is

$$y''' = \frac{3y'y''^2}{1 + y'^2}. \tag{27}$$

Equating this value of y''' to the value of y''' in the equation of the tautochrones (5), and reducing, we find

$$\left.\begin{aligned}(\phi + \psi y')y''^2 - [\phi_y + 2\psi_x + 3(\psi_y - \phi_x)y' - (2\phi_y + \psi_x)y'^2](1 + y'^2)y''\\ - [\phi_{xx} + (2\phi_{xy} + \psi_{xx})y' + (\phi_{yy} + 2\psi_{xy})y'^2 + \psi_{yy}y'^3](1 + y'^2)^2 = 0,\end{aligned}\right\} \tag{28}$$

which is the condition for hyperosculating circles.

This is a quadratic in y'', regarding the point (x, y) and the direction y' as given. Hence there are two values of y'' for each lineal element in the plane, corresponding to which two tautochrones are hyperosculated by their circles of curvature. Thus, with each lineal element through a given point we may associate two points, namely the centers of the two hyperosculating circles determined by this element. We now inquire what will be the locus of these centers as the element rotates about the given point; that is, as y' varies in equation (28).

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[*] That is, a force whose rectangular components ϕ, ψ are conjugate harmonic functions, so named by Lecornu, "Sur les forces analytiques," *Journal de l'École Polytechnique*, Vol. LV (1885).

If we represent by $\bar{X}$ and $\bar{Y}$ the coordinates of the center of curvature referred to the given point (x, y) as origin, we have the formulas:

$$\bar{X} = -\frac{y'(1+y'^2)}{y''}, \quad \bar{Y} = \frac{1+y'^2}{y''}. \tag{29}$$

Substituting in equation (28) the values of y' and y'' found from these equations, we obtain

$$\left.\begin{aligned} \phi_{xx}\bar{Y}^3 - (2\phi_{xy}+\psi_{xx})\bar{X}\bar{Y}^2 + (\phi_{yy}+2\psi_{xy})\bar{X}^2\bar{Y} - \psi_{yy}\bar{X}^3 \\ + (\phi_y + 2\psi_x)\bar{Y}^2 - 3(\psi_y - \phi_x)\bar{X}\bar{Y} - (2\phi_y+\psi_x)\bar{X}^2 - \phi\bar{Y} + \psi\bar{X} = 0. \end{aligned}\right\} \tag{30}$$

This represents the locus of the centers of the hyperosculating circles, $\bar{X}$, $\bar{Y}$ being the coordinates of a point on the locus referred to the given point (x, y) as origin. The coefficients are determined by the coordinates of the given point. The curve is a cubic through the origin. The tangent at the origin is

$$\bar{Y} = \frac{\psi}{\phi}\bar{X}; \tag{31}$$

hence it is the line along which the force acts at that point.

THEOREM IV. *Corresponding to each lineal element through a given point there are two tautochrones tangent to the element at that point which are hyperosculated by their circles of curvature. The locus of the centers of these hyperosculating circles as the slope of the element changes is a cubic through the given point. The tangent to the cubic at the given point is the line along which the force acts at that point.*

10. We have found that the triply infinite system of tautochrones represented by equation (5) is such that the singly infinite system of them, passing through any point in the plane in a given direction, possesses the properties stated in Theorems I and II, and the doubly infinite system of them, passing through any point in the plane, possesses the properties stated in Theorems III and IV. We shall express this by saying that the triply infinite system possesses properties I, II, III and IV.

Converse Questions.

11. We have shown that the triply infinite system of tautochrones in the plane possesses properties I, II, III and IV. The question now arises, does any triply infinite system of curves possessing properties I–IV represent a family of tautochrones? We now proceed to convert these four properties in succession.

12. *Conversion of property I.* The focal curve of any system of curves having property I will be of the form

$$(\alpha^2+\beta^2)^2+A_1(\alpha y'-\beta)(\alpha^2+\beta^2)+(A_2\alpha+A_3\beta)(\alpha y'-\beta)=0. \tag{32}$$

This is the general equation of a bicircular quartic having a flecnode at the origin, the stationary tangent at the origin having the direction y', where A_1, A_2, A_3 are arbitrary functions of x, y and y'.

We now eliminate α and β between equations (8) and (32) by means of relation (9). This gives

$$y'''=B_1+B_2y''+B_3y''^2, \tag{33}$$

where B_1, B_2, B_3 involve A_1, A_2, A_3 and are functions of x, y and y'. Thus all systems possessing property I are of the form (33).

13. *Conversion of property II.* In order to convert Theorem II, we state it in a form not assuming a field of force, as follows: At any point (x, y) of the plane there exists a certain direction, whose slope is $\omega(x, y)$, such that the tangent of the angle between the two tangents t_1 and t_2 to the focal quartic at the given point is three times the tangent of the angle between t_1 and this direction. We now seek the equation of the system possessing property II in addition to property I.

The focal curve of the general system (33) possessing property I is

$$\left.\begin{array}{c}4B_1(\alpha^2+\beta^2)^2-2B_2(\alpha y'-\beta)(\alpha^2+\beta^2)\\+(\alpha y'-\beta)\{[B_3y'(y'^2+1)-3(y'^2-1)]\alpha-[B_3(y'^2+1)-6y']\beta\}=0.\end{array}\right\} \tag{34}$$

The slopes of the tangents t_1 and t_2 at the origin are respectively

$$\frac{\beta}{\alpha}=y', \quad \frac{B_3y'(y'^2+1)-3(y'^2-1)}{B_3(y'^2+1)-6y'}. \tag{35}$$

If the system possesses property II, we then have

$$\frac{\dfrac{B_3 y'(y'^2+1)-3(y'^2-1)}{B_3(y'^2+1)-6y'}-y'}{1+\dfrac{B_3 y'(y'^2+1)-3(y'^2-1)}{B_3(y'^2+1)-6y'}y'}=\frac{3(\omega-y')}{1+\omega y'}. \tag{36}$$

Solving for B_3, we find

$$B_3=\frac{1+4\omega y'-3y'^2}{(\omega-y')(y'^2+1)}. \tag{37}$$

Hence all systems possessing both properties I and II have the form

$$y'''=B_1+B_2 y''+\frac{1+4\omega y'-3y'^2}{(\omega-y')(1+y'^2)}y''^2. \tag{38}$$

14. *Conversion of property III.* We may state property III in a form not assuming a field of force, as follows: Corresponding to each lineal element through a given point there is a focal quartic whose inverse with respect to the given point is a hyperbola. The locus of the center of this hyperbola as y' varies is a circular cubic having the given point as double point; the real asymptote of the cubic is parallel to the direction ω introduced in property II. We now determine the equation of the system having property III in addition to properties I and II.

The focal curve of the system (38) may be obtained from (34) by substituting in it the value of B_3 given in (37). Then inverting this focal curve by means of formulas (16), we obtain the hyperbola

$$\left.\begin{aligned}&[(3\omega-2y'+\omega y'^2)y']\xi^2-[(3\omega-2y'+\omega y'^2)+(1-2\omega y'+3y'^2)y']\xi\eta\\&\quad+[1-2\omega y'+3y'^2]\eta^2-2B_2(\xi y'-\eta)(\omega-y')+4B_1(\omega-y')=0.\end{aligned}\right\} \tag{39}$$

If property III holds for this system, the center of the hyperbola (39) must lie on the cubic

$$(\omega X-Y)(X^2+Y^2)+H_1X^2+H_2XY+H_3Y^2=0, \tag{40}$$

where H_1, H_2 and H_3 are arbitrary functions of x and y. Computing the coordinates of the center of the hyperbola, making use of relation (20), we find

$$X=\frac{2B_2}{3(1+y'^2)},\quad Y=\frac{2B_2y'}{3(1+y'^2)}. \tag{41}$$

Substituting in (40), we get

$$B_2 = -\frac{3(H_1 + H_2 y' + H_3 y'^2)}{2(\omega - y')} = \frac{L_1 + L_2 y' + L_3 y'^2}{\omega - y'}, \tag{42}$$

where L_1, L_2 and L_3 are functions of x and y. Hence all systems having properties I, II and III are of the form

$$y''' = B_1 + \frac{L_1 + L_2 y' + L_3 y'^2}{\omega - y'} y'' + \frac{1 + 4\omega y' - 3y'^2}{(\omega - y')(1 + y'^2)} y''^2. \tag{43}$$

15. *Conversion of property IV.* Finally, we inquire what systems possess property IV in addition to properties I, II and III. Property IV may be stated without reference to force, as follows: For a system of curves through a given point in the plane, the locus of the centers of the hyperosculating circles constructed at the given point is a cubic through that point, whose tangent at the given point has the slope ω introduced in property II.

If the curves of the system (43) are to possess property IV, the centers of their hyperosculating circles must lie on the cubic,

$$M_1\bar{Y}^3 + M_2\bar{X}\bar{Y}^2 + M_3\bar{X}^2\bar{Y} + M_4\bar{X}^3 + N_1\bar{Y}^2 + N_2\bar{X}\bar{Y} + N_3\bar{X}^2 + \bar{Y} - \omega\bar{X} = 0, \tag{44}$$

where the M's and N's are functions of x and y. Equating the values of y''' in equations (43) and (27) gives the equation of condition for hyperosculation. If, in this equation, we substitute the values of y' and y'' found from (29), we obtain the locus of the centers of the hyperosculating circles for system (43), namely

$$\frac{B_1(\omega\bar{Y} + \bar{X})\bar{Y}^4}{\bar{X}^2 + \bar{Y}^2} + L_1\bar{Y}^2 - L_2\bar{X}\bar{Y} + L_3\bar{X}^2 + \bar{Y} - \omega\bar{X} = 0. \tag{45}$$

If this equation is to be of the form (44), we find

$$B_1 = \frac{(K_1 + K_2 y' + K_3 y'^2 + K_4 y'^3)(1 + y'^2)}{\omega - y'}, \tag{46}$$

where the K's are functions of x and y; so that we have finally this result:

The general equation of the system of curves having properties I, II, III and IV is

$$\left.\begin{aligned} y''' = {} & \frac{(K_1 + K_2 y' + K_3 y'^2 + K_4 y'^3)(1 + y'^2)}{\omega - y'} \\ & + \frac{L_1 + L_2 y' + L_3 y'^2}{\omega - y'} y'' + \frac{1 + 4\omega y' - 3y'^2}{(\omega - y')(1 + y'^2)} y''^2. \end{aligned}\right\} \tag{47}$$

This equation contains eight arbitrary functions of x and y, namely K_1, K_2, K_3, K_4, L_1, L_2, L_3 and ω. It is of the same form in y', y'' and y''' as equation (5), but more general with respect to the functions of x and y which enter into the coefficients. Properties I–IV are, then, not sufficient to characterize the total triply infinite system of tautochrones in the plane (5), since these properties belong to the more general system (47). In the next section we shall find the relations which must hold between the eight functions K_1, K_2, K_3, K_4, L_1, L_2, L_3 and ω for a complete characterization.

If, however, we consider only the doubly infinite system of curves passing through a point (x, y) and possessing properties I–IV, the curves of this system will have at that point the same values of y', y'' and y''' as the curves of the doubly infinite system of tautochrones passing through that point; that is, properties I–IV characterize the system of tautochrones through a given point up to differential elements of the third order. This can be shown in the following way:

For a given point, the eight arbitrary functions of x and y in equation (47) reduce to constants. If, now, these eight functions are chosen as arbitrary constants, it is required to find functions ϕ, ψ which for a given value of x and y make equation (5) equivalent to equation (47); that is, we must find functions ϕ, ψ which satisfy the following set of equations obtained by equating the coefficients in equations (5) and (47):

$$\left.\begin{aligned}
K_1 &= -\frac{\phi_{xx}}{\phi}, & L_1 &= \frac{\phi_y + 2\psi_x}{\phi},\\
K_2 &= -\frac{2\phi_{xy} + \psi_{xx}}{\phi}, & L_2 &= \frac{3(\phi_x - \psi_y)}{\phi},\\
K_3 &= -\frac{\phi_{yy} + 2\psi_{xy}}{\phi}, & L_3 &= \frac{2\phi_y + \psi_x}{\phi},\\
K_4 &= -\frac{\psi_{yy}}{\phi}, & \omega &= \frac{\psi}{\phi},
\end{aligned}\right\} \qquad (48)$$

in which K_1, K_2, K_3, K_4, L_1, L_2, L_3 and ω are arbitrary constants. Let

$$\left.\begin{aligned}
\phi &= a + bx + cy + dxy + ex^2 + fy^2,\\
\psi &= a' + b'x + c'y + d'xy + e'x^2 + f'y^2.
\end{aligned}\right\} \qquad (49)$$

Then for a given point, say $x = y = 0$, we have

$$\left.\begin{array}{llll} \phi \ = a, & \psi \ = a', & \phi_x = b, & \phi_y = c, \\ \psi_x = b' & \psi_y = c', & \phi_{xx} = 2e, & \phi_{xy} = d, \\ \phi_{yy} = 2f, & \psi_{xx} = 2e' & \psi_{xy} = d', & \psi_{yy} = 2f'. \end{array}\right\} \quad (50)$$

Then equations (48) reduce to

$$\left.\begin{array}{ll} aK_1 = -2e, & aL_1 = -(c + 2b'), \\ aK_2 = -2(d + e'), & aL_2 = 3(b - c'), \\ aK_3 = -2(f + d'), & aL_3 = 2c + b', \\ aK \ = -2f', & a\omega \ = a'. \end{array}\right\} \quad (51)$$

Take $a = 1$, $c' = d' = e' = 0$. Then

$$\left.\begin{array}{llll} e = -K_1/2, & d = -K_2/2, & f = -K_3/2, & f' = -K_4/2, \\ c = (L_1 + 2L_3)/3, & b = L_2/3, & b' = -(2L_1 + L_3)/3, & a' = \omega. \end{array}\right\} \quad (52)$$

Hence, from (49) and (52),

$$\left.\begin{array}{l} \phi = 1 + \dfrac{L_3}{2}x + \dfrac{L_1 + 2L_3}{3}y - \dfrac{K_2}{2}xy - \dfrac{K_1}{2}x^2 - \dfrac{K_3}{2}y^2, \\ \psi = \omega - \dfrac{2L_1 + L_3}{3}x - \dfrac{K_4}{2}y^2. \end{array}\right\} \quad (53)$$

The system of curves, possessing properties I–IV and passing through the point (0, 0), coincides up to differential elements of the third order with the system of tautochrones due to a force whose components are given by (53). The particular pair of functions (53) is only one of many pairs that satisfy the requirement. Properties I–IV are therefore sufficient to characterize the doubly infinite system of tautochrones through a given point up to differential elements of the third order.

Complete Characterization.

16. If the triply infinite system of curves represented by equation (47) is to be a system of tautochrones, it must be identical with the system represented by equation (5). The set of relations (48) must therefore hold, where now K_1, K_2, K_3, K_4, L_1, L_2, L_3 and ω represent arbitrary functions of x and y.

If we introduce

$$h = \log \phi, \qquad (54)$$

and substitute

$$\psi = \omega \phi \qquad (55)$$

in the equations for L_1, L_2 and L_3 in (48), we find

$$L_1 = -h_y - 2\omega h_x - 2\omega_x, \tag{56}$$

$$L_2 = 3(h_x - \omega h_y - \omega_y), \tag{57}$$

$$L_3 = 2h_y + \omega h_x + \omega_x. \tag{58}$$

From (56) and (58),

$$h_x = -\frac{3\omega_x + 2L_1 + L_3}{3\omega}, \quad h_y = \frac{L_1 + 2L_3}{3}. \tag{59}$$

Substituting these values in (57), we get the relation

$$\omega^2(L_1 + 2L_3) + \omega L_2 + 3(\omega_x + \omega\omega_y) + 2L_1 + L_3 = 0. \tag{A}$$

Also, from (59),

$$(L_1 + 2L_3)_x + \left(\frac{3\omega_x + 2L_1 + L_3}{\omega}\right)_y = 0. \tag{B}$$

By virtue of these two relations (A) and (B), we can express L_1, L_2, L_3 and ω in terms of two arbitrary functions ϕ and ψ; for, from (B) we can find a function h satisfying equations (59), thus giving (56) and (58); then (57) follows from (A), and, making the substitutions (54) and (55), L_1, L_2, L_3 and ω are obtained in the form given in (48).

Again, from (48),

$$\phi K_2 = -(\phi L_3)_x, \quad \phi K_3 = (\phi L_1)_y, \tag{60}$$

from which we get the two relations giving K_2 and K_3:

$$K_2 = -(L_3)_x + \frac{L_3(3\omega_x + 2L_1 + L_3)}{3\omega}, \tag{C}$$

$$K_3 = (L_1)_y + \frac{L_1(L_1 + 2L_3)}{3}. \tag{D}$$

To obtain the equations determining K_1, we make use of h_{xx}, found from (55),

$$h_{xx} = \frac{\phi_{xx}}{\phi} - h_x^2. \tag{61}$$

Then, from (61) and (59), the equation for K_1 (48) gives

$$K_1 = -\left(\frac{3\omega_x + 2L_1 + L_3}{3\omega}\right)^2 + \left(\frac{3\omega_x + 2L_1 + L_3}{3\omega}\right)_x. \tag{E}$$

Finally, to obtain the equation for K_4, from (49) we find the relation

$$\frac{\psi_y}{\phi} = \omega h_y + \omega_y. \tag{62}$$

Differentiating both sides of this equation with respect to y, and substituting for ψ_{yy}/ϕ its value, $-K_4$, from (48), we get

$$K_4 = -\omega\left(\frac{L_1+2L_3}{3}\right)^2 - 2\omega_y\frac{L_1+2L_3}{3} - \frac{\omega\,[(L_1)_y+2(L_3)_y]}{3} - \omega_{yy}. \qquad \text{(F)}$$

If, then, the relations (C), (D), (E) and (F) hold in addition to (A) and (B), they are sufficient for determining, in terms of two arbitrary functions ϕ and ψ, the values of K_1, K_2, K_3, K_4, L_1, L_2, L_3 and ω given in equations (48). We then have the final result:

Any triply infinite system of curves in the plane represents a family of tautochrones when, and only when, it possesses the properties I, II, III and IV, and the functions K_1, K_2, K_3, K_4, L_1, L_2, L_3 *and* ω, *which appear in the equation* (47) *of the system, satisfy the relations* (*A*), (*B*), (*C*), (*D*), (*E*) *and* (*F*).

Point Transformations.

17. *The formulas of transformation.* We shall refer to systems of curves possessing properties I; I and II; I, II and III; and I, II, III and IV as systems of type I, II, III and IV respectively. The equations of these systems are respectively (33), (38), (43) and (47). We now consider the effect of an arbitrary point transformation on such systems.

We write the transformation formulas as follows, using the notation of Kasner,*

$$X = \Phi(x, y), \qquad Y = \Psi(x, y), \qquad (63)$$

$$(c\,y' + d)\,Y' = a\,y' + b, \qquad (64)$$

where

$$a = \Psi_y, \quad b = \Psi_x, \quad c = \Phi_y, \quad d = \Phi_x.$$

$$(c\,y' + d)^3\,Y'' = \alpha + \beta\,y' + \gamma\,y'^2 + \delta\,y'^3 + j\,y'', \qquad (65)$$

where

$$\begin{aligned}
j &= a\,d - b\,c,\\
\alpha &= \Phi_x\Psi_{xx} - \Psi_x\Phi_{xx},\\
\beta &= \Phi_y\Psi_{xx} - \Psi_y\Phi_{xx} + 2\,\Phi_x\Psi_{xy} - 2\,\Psi_x\Phi_{xy},\\
\gamma &= \Phi_x\Psi_{yy} - \Psi_x\Phi_{yy} + 2\,\Phi_y\Psi_{xy} - 2\,\Psi_y\Phi_{xy},\\
\delta &= \Phi_y\Psi_{yy} - \Psi_y\Phi_{yy}.
\end{aligned}$$

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* Kasner: "Trajectories," *loc. cit.*, p. 419.

We shall use (65) in the abbreviated form

$$Y'' = \varepsilon_0 + \varepsilon_1 y''. \tag{65'}$$

$$\left.\begin{aligned}(c y' + d)^5 Y''' = {} & (c y' + d)(A_0 + A_1 y' + A_2 y'^2 + A_3 y'^3 + A_4 y'^4) \\ & - 3(\alpha + \beta y' + \gamma y'^2 + \delta y'^3)(\alpha_2 + \beta_2 y' + \gamma_2 y'^2) \\ & + (\alpha_1 + \beta_1 y' + \gamma_1 y'^2) y'' - 3 c j y''^2 + j(c y' + d) y''',\end{aligned}\right\} \quad (66)$$

where

$$\begin{gathered}\alpha_1 = d(\beta + j_x) - 3 c \alpha - 3 j d_x, \\ \beta_1 = c(\beta + j_x) + d(2\gamma + j_y) - 3 c \beta - 3 j (c_x + d_y), \\ \gamma_1 = c(2\gamma + j_y) + 3 d \delta - 3 c \gamma - 3 j c_y, \\ \alpha_2 = \Phi_{xx}, \quad \beta_2 = 2\Phi_{xy}, \quad \gamma_2 = \Phi_{yy}, \\ A_0 = \alpha_x, \quad A_1 = \alpha_y + \beta_x, \quad A_2 = \beta_y + \gamma_x, \quad A_3 = \gamma_y + \delta_x, \quad A_4 = \delta_y.\end{gathered}$$

We shall use (66) in the abbreviated form

$$Y''' = E_0 + E_1 y'' + E_2 y''^2 + E_3 y'''. \tag{66'}$$

18. *Transformation of systems of type I.* If we apply the above transformation to the equation of systems of type I,

$$y''' = B_1 + B_2 y'' + B_3 y''^2, \tag{33}$$

it becomes

$$y''' = f_1 + f_2 y'' + f_3 y''^2, \tag{67}$$

where

$$f_1 = \frac{\overline{B}_1 + \overline{B}_2 \varepsilon_0 + \overline{B}_3 \varepsilon_0^2 - E_0}{E_3}, \quad f_2 = \frac{\overline{B}_2 \varepsilon_1 + 2 \overline{B}_3 \varepsilon_0 \varepsilon_1 - E_1}{E_3}, \quad f_3 = \frac{\overline{B}_3 \varepsilon_1^2 - E_2}{E_3},$$

the bars denoting the result of applying the transformation to the quantities under them. The f's, like the B's, are functions of x, y and y', so that (67) is of the same form as (33).

Systems of type I are converted into systems of the same type by the general point transformation.

19. *Transformation of systems of type II.* For a system of type II, equation (38), we have

$$B_3 = \frac{1 + 4\omega y' - 3 y'^2}{(\omega - y')(1 + y'^2)}.$$

If this system is to remain of the same type, we must have

$$f_3 = \frac{\bar{B}_3 \varepsilon_1^2 - E_2}{E_3} = \frac{1 + 4\,\Omega\, y' - 3\, y'^2}{(\Omega - y')(1 + y'^2)}, \tag{68}$$

where Ω is a function of x and y.

Applying the transformation of Article 16 to B_3, we find

$$\bar{B}_3 = \frac{(c y' + d)\left[(c y' + d)^2 + 4\,\bar{\omega}\,(a y' + b)(c y' + d) - (a y' + b)^2\right]}{\left[(c y' + d)\,\bar{\omega} - (a y' + b)\right]\left[(a y' + b)^2 + (c y' + d)^2\right]}. \tag{69}$$

Then the value of f_3 becomes

$$f_3 = \frac{N}{D}, \tag{70}$$

$$\begin{aligned} N = {} & (4\,a\,b\,d - b^2 c + 3\,c\,d^2)\,\bar{\omega} - (4\,b\,c\,d - a\,d^2 + 3\,a\,b^2) \\ & + 2\left[(2a^2 d + a b c + 3c^2 d)\,\bar{\omega} - (2bc^2 + a c d + 3a^2 b)\right] y' + 3(c\,\bar{\omega} - a)(a^2 + c^2)\, y'^2, \\ D = {} & \left[d\,\bar{\omega} - b + (c\,\bar{\omega} - a)\, y'\right]\left[b^2 + d^2 + 2\,(a\,b + c\,d)\, y' + (a^2 + c^2)\, y'^2\right]. \end{aligned}$$

If this is to be of the form (63), we must have

$$a\,b + c\,d = 0, \quad a^2 + c^2 = b^2 + d^2, \tag{71}$$

in which case (70) reduces to

$$\left.\begin{aligned} f_3 &= \frac{1 + 4\,\Omega\, y' - 3\, y'^2}{(\Omega - y')(1 + y'^2)}, \\ \Omega &= \frac{d\,\bar{\omega} - b}{a - c\,\bar{\omega}}. \end{aligned}\right\} \tag{72}$$

Equations (71), written in the Φ, Ψ notation, are

$$\left.\begin{aligned} \Phi_x \Phi_y + \Psi_x \Psi_y &= 0, \\ \Phi_x^2 + \Psi_x^2 &= \Phi_y^2 + \Psi_y^2. \end{aligned}\right\} \tag{71'}$$

Eliminating Ψ_y, we find

$$(\Phi_x^2 + \Psi_x^2)(\Phi_y^2 - \Psi_x^2) = 0.$$

Disregarding the factor $\Phi_x^2 + \Psi_x^2$, since it leads only to imaginary solutions or to the trivial solution, $\Phi_x = \Psi_x = \Phi_y = \Psi_y = 0$, which makes Φ and Ψ constants and throws the whole plane into a single point, we have for the solutions of equations (71′),

$$\left.\begin{aligned} \Phi_x &= \Psi_y, \\ \Phi_y &= -\Psi_x, \end{aligned} \quad \text{or} \quad \begin{aligned} \Phi_x &= -\Psi_y, \\ \Phi_y &= \Psi_x. \end{aligned}\right\} \tag{73}$$

These are respectively the direct and inverse conformal transformations.

The only point transformations under which systems of type II are converted into systems of the same type are the conformal transformations.

20. *Transformation of systems of types III and IV.* For a system of type III, equation (43), we have

$$B_2 = \frac{L_1 + L_2 y' + L_3 y'^2}{\omega - y'}.$$

If this system is to remain of the same type, we must have, in addition to (68),

$$f_2 = \frac{B_3 \varepsilon_1 + 2 B_3 \varepsilon_0 \varepsilon_1 - E_1}{E_3} = \frac{M_1 + M_2 y' + M_3 y'^2}{\Omega - y'}, \tag{74}$$

where M_1, M_2 and M_3 are functions of x and y.

Under the transformation of Article 17, B_2 becomes

$$\bar{B}_2 = \frac{\bar{L}_1 (c y' + d)^2 + \bar{L}_2 (a y' + b)(c y' + d) + \bar{L}_3 (a y' + b)^2}{(c y' + d)\left[(c y' + d)\bar{\omega} - (a y' + b)\right]}.$$

It is found, by forming the expression for f_2 and imposing on it the condition (74), that this leads to the necessary condition

$$\alpha = \beta = \gamma = \delta = 0. \tag{75}$$

Condition (75), expressed in the Φ, Ψ notation, is

$$\left.\begin{aligned} &\Phi_x \Psi_{xx} - \Psi_x \Phi_{xx} = 0, \\ &\Phi_y \Psi_{xx} - \Psi_y \Phi_{xx} + 2 \Phi_x \Psi_{xy} - 2 \Psi_x \Phi_{xy} = 0, \\ &\Phi_x \Psi_{yy} - \Psi_x \Phi_{yy} + 2 \Phi_y \Psi_{xy} - 2 \Psi_y \Phi_{xy} = 0, \\ &\Phi_y \Psi_{yy} - \Psi_y \Phi_{yy} = 0. \end{aligned}\right\} \tag{75'}$$

The solution of this system of equations is*

$$\Phi = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3}, \quad \Psi = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}. \tag{76}$$

This is the projective transformation. If, then, a system of type III is to be converted into a system of the same type, the transformation must be projective as well as conformal. The most general transformation of this kind is the similitude transformation,

$$\begin{aligned} \Phi &= a x + b y + c, \\ \Psi &= \mp b x \pm a y + d, \end{aligned} \tag{77}$$

where a, b, c and d are constants. Under this transformation the condition (74) is satisfied. It is also found, by computing $\overline{B}_1$ and f_1, that systems of type IV remain of the same type under the transformation (77), so that we have finally the result:

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* Lie-Scheffers: "Vorlesungen über continuierliche Gruppen," p. 34.

The only point transformations which convert systems of type III and IV into systems of the same type are the four-parameter group of similitude transformations.

Systems of Tautochrones for a Given k.

21. *The differential equation of the system.* The total triply infinite family of tautochrones in the plane may be grouped into doubly infinite systems according to different values of the time constant k. From equation (1) we have for the equation of such a system

$$T_s = k. \tag{78}$$

For each value of k this equation represents a doubly infinite system of curves.

From equation (4) we find

$$T_s = \overline{T} + \frac{N}{r}, \tag{79}$$

where

$$\overline{T} = \frac{\phi_x + (\phi_y + \psi_x)\, y' + \psi_y\, y'^2}{1 + y'^2}.$$

Hence the differential equation of the system (73), expressed in Cartesian form, is

$$\frac{\phi_x + (\phi_y + \psi_x)\, y' + \psi_y\, y'^2}{1 + y'^2} + \frac{(\psi - y'\phi)y''}{(1 + y'^2)^2} = k. \tag{80}$$

22. *Geometric properties of the system.* At every point (x, y) of the plane in a given direction y' and corresponding to a given value of k there is one tautochrone. Suppose that at the given point (x, y) the osculating circle of this tautochrone is constructed. Let α, β denote the coordinates of the center of this osculating circle, referred to the point (x, y) as origin. We inquire what will be the locus of this point (α, β) as y' varies.

The formulas for the coordinates of the center of curvature are

$$\alpha = -\frac{y'(1 + y'^2)}{y''}, \quad \beta = \frac{1 + y'^2}{y''}. \tag{81}$$

Eliminating y' and y'' between equations (80) and (81), we obtain

$$\alpha^2 (k - \psi_y) + \beta^2 (k - \phi_x) + (\phi_y + \psi_x)\, \alpha\beta - \phi\alpha - \psi\beta = 0. \tag{82}$$

This equation represents the locus of the centers of the osculating circles drawn at the point (x, y) to the tautochrones which pass through this point and correspond to a given value of the time constant k, α and β being the coordinates of a point of the locus referred to the given point (x, y) as origin. The locus is a conic passing through the given point. The tangent to the conic at the given point is

$$\beta = -\frac{\phi}{\psi}\alpha; \tag{83}$$

hence its direction is perpendicular to the direction in which the force is acting at the point.

In the case of gravity, $\phi = 0$, $\psi = g$, and the conic reduces to

$$k(\alpha^2 + \beta^2) - \beta = 0, \tag{84}$$

a circle tangent to the α axis at the origin.

It is also found that as k varies, the locus of the center of the above conic is another conic through the given point (x, y). The line along which the force acts at the given point is tangent to the second conic.

Space Tautochrones.

23. *The differential equation of the system of tautochrones in three-dimensional space.* Suppose that a positional force $F(x, y, z)$ is acting at every point of three-dimensional space. We shall now consider the system of space tautochrones due to such a force. The physical property of the tautochrones, expressed in equation (1), yields as in the plane case,

$$T_{ss} = 0, \tag{85}$$

where T is the tangential component of the force F along the tautochrone. We wish to express this equation in Cartesian form.

Let N represent the normal component of the force F, and ϕ, ψ and χ its x, y and z components respectively. Denoting the direction cosines of the tangent and principal normal by α, β, γ and l, m, n respectively, we have the relations

$$\left.\begin{aligned} \alpha, \beta, \gamma &= dx, dy, dz/ds = 1, y', z'/\sqrt{1 + y'^2 + z'^2}, \\ l, m, n &= \frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}, \frac{d^2z}{ds^2} \bigg/ \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}. \end{aligned}\right\} \tag{86}$$

Then we find

$$T = \phi\alpha + \psi\beta + \chi\gamma = \frac{\phi + \psi y' + \chi z'}{\sqrt{1 + y'^2 + z'^2}}, \tag{87}$$

$$N = \phi l + \psi m + \chi n = \left(\phi\frac{d^2x}{ds^2} + \psi\frac{d^2y}{ds^2} + \chi\frac{d^2z}{ds^2}\right) r, \tag{88}$$

where r, the radius of absolute curvature, is

$$r = 1 \Big/ \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}.$$

From equation (88) we get by means of relations (86)

$$\frac{N}{r} = \frac{\psi y'' + \chi z''}{1 + y'^2 + z'^2} - \frac{(\phi + \psi y' + \chi z')(y'y'' + z'z'')}{(1 + y'^2 + z'^2)^2}. \tag{89}$$

Differentiating equation (87) with respect to s, we find, after reduction,

$$T_s = \bar{T} + \frac{N}{r}, \tag{90}$$

where

$$\bar{T} = \frac{\phi_x + \phi_y y' + \phi_z z' + (\psi_x + \psi_y y' + \psi_z z')\,y' + (\chi_x + \chi_y y' + \chi_z z')\,z'}{1 + y'^2 + z'^2}. \tag{91}$$

Differentiating equation (90) with respect to s gives the intrinsic equation of the tautochrones

$$\bar{T}_s + \left(\frac{N}{r}\right)_s = 0. \tag{92}$$

If the values of $\bar{T}_s$ and $\left(\frac{N}{r}\right)_s$ are found from equations (91) and (89) by differentiation and substituted in (92), we obtain finally

$$H y''' + K z''' = A + B y'' + C z'' + D y'' z'' + E y''^2 + F z''^2, \tag{93}$$

where

$$H = \phi y' - \psi(1 + z'^2) + \chi y' z',$$

$$K = \phi z' - \chi(1 + y'^2) + \psi y' z',$$

$$\begin{aligned} A = {} & [\phi_{xx} + (2\phi_{xy} + \psi_{xx})\,y' + (2\phi_{xz} + \chi_{xx})\,z' + 2(\phi_{yz} + \psi_{xz} + \chi_{xy})\,y' z' \\ & + (\phi_{yy} + 2\psi_{xy})\,y'^2 + (\phi_{zz} + 2\chi_{xz})\,z'^2 + (\psi_{zz} + 2\chi_{yz})\,y' z'^2 \\ & + (2\psi_{yz} + \chi_{yy})\,y'^2 z' + \psi_{yy}\,y'^3 + \chi_{zz}\,z'^3]\,(1 + y'^2 + z'^2), \end{aligned}$$

$$\begin{aligned} B = {} & \phi_y + 2\psi_x + 3(\psi_y - \phi_x)\,y' + (\chi_y + 2\psi_z)\,z' - 3(\phi_z + \chi_x)\,y' z' \\ & - (2\phi_y + \psi_x)\,y'^2 + (\phi_y + 2\psi_x)\,z'^2 + 3(\psi_y - \chi_z)\,y' z'^2 \\ & - (2\chi_y + \psi_z)\,y'^2 z' + (\chi_y + 2\psi_z)\,z'^3, \end{aligned}$$

$$
\begin{aligned}
C = {} & \phi_z + 2\chi_x + (\psi_z + 2\chi_y)\,y' + 3\,(\chi_z - \phi_x)\,z' - 3\,(\phi_y + \psi_x)\,y'z' \\
& + (\phi_z + 2\chi_x)\,y'^2 - (2\phi_z + \chi_x)\,z'^2 - (2\psi_z + \chi_y)\,y'z'^2 \\
& + 3\,(\chi_z - \psi_y)\,y'^2 z' + (\psi_z + 2\chi_y)\,y'^3,
\end{aligned}
$$

$$D = -\frac{3\,(\psi z' + \chi y') - [5\,(\psi y' + \chi z') + 8\,\phi]\,y'z' + 3\,(\psi z'^3 + \chi y'^3)}{1 + y'^2 + z'^2},$$

$$E = \frac{3\,\phi y'^2 - \phi - 4\,\psi y' + (3\,\chi y'^2 - \chi)\,z' - (\phi + 4\,\psi y')\,z'^2 - \chi z'^3}{1 + y'^2 + z'^2},$$

$$F = \frac{3\,\phi z'^2 - \phi - 4\,\chi z' + (3\,\psi z'^2 - \psi)\,y' - (\phi + 4\,\chi z')\,y'^2 - \psi y'^3}{1 + y'^2 + z'^2}.$$

Equation (93) is the differential equation of the totality of tautochrones in three-dimensional space, due to a positional force whose rectangular components are ϕ, ψ and χ. It represents a system of ∞^∞ curves. When χ and z are absent, the equation becomes that of the plane system, equation (5).

24. *Condition that every straight line in space shall be a tautochrone.* In order that every straight line in space shall be a tautochrone, it is necessary and sufficient that the force be such that equation (93) is satisfied when $y'' = z'' = 0$. Hence the coefficient A must vanish identically. This condition leads to the following system of equations:

$$
\left.
\begin{aligned}
\phi_{xx} &= 0, & 2\,\phi_{xz} + \chi_{xx} &= 0, \\
\psi_{yy} &= 0, & \phi_{zz} + 2\,\chi_{xz} &= 0, \\
\chi_{zz} &= 0, & 2\,\psi_{yz} + \chi_{yy} &= 0, \\
2\,\phi_{xy} + \psi_{xx} &= 0, & \psi_{zz} + 2\,\chi_{yz} &= 0, \\
\phi_{yy} + 2\,\psi_{xy} &= 0, & \phi_{yz} + \psi_{xz} + \chi_{xy} &= 0.
\end{aligned}
\right\} \quad (94)
$$

The solution of this set of equations may be obtained without difficulty. From the first three we have

$$
\left.
\begin{aligned}
\phi &= f(y, z) + x\,F(y, z), \\
\psi &= g\,(x, z) + y\,G\,(x, z), \\
\chi &= h\,(x, y) + z\,H(x, y),
\end{aligned}
\right\} \quad (95)
$$

where f, g, h, F, G, H are unknown functions. The remaining equations of (94) serve to determine these functions and we obtain for the solution of the system (94),

$$
\left.
\begin{aligned}
\phi &= a + b\,x + c\,y + d\,z + e\,x\,y + f\,x\,z + g\,y\,z + h\,y^2 + i\,z^2, \\
\psi &= a' + b'\,x + c'\,y + d'\,z - h\,x\,y + f'\,x\,z + g'\,y\,z - e\,x^2 + i'\,z^2, \\
\chi &= a'' + b''\,x + c''\,y + d''\,z - (f' + g)\,x\,y - i\,x\,z - i'\,y\,z - f\,x^2 - g'\,y^2,
\end{aligned}
\right\} \quad (96)
$$

where the coefficients are constants.

The class of forces defined by equations (96) is the class for which every straight line in space is a tautochrone. Equations (96) involve twenty parameters. The class of collineations involves fifteen parameters. Hence the relation, pointed out by Kasner,* which exists between these two classes in the plane, does not hold in space of three dimensions.

If the class of forces defined by equations (96) is required to be conservative, then

$$\phi_y = \psi_x, \quad \phi_z = \chi_x, \quad \psi_z = \chi_y, \tag{97}$$

in which case all the second partial derivatives of ϕ, ψ and χ vanish; the force components have the linear form,

$$\left.\begin{aligned} \phi &= a + bx + cy + dz, \\ \psi &= a' + cx + c'y + d'z, \\ \chi &= a'' + dx + d'y + d''z, \end{aligned}\right\} \tag{98}$$

and the work function is

$$W = ax + a'y + a''z + cxy + dxz + d'yz + \frac{bx^2}{2} + \frac{c'y^2}{2} + \frac{d''z^2}{2}. \tag{99}$$

Hence the result stated in the paper cited above for plane fields of force holds in space of three dimensions: *The only conservative forces for which every straight line is a tautochrone are those in which the potential is a quadratic function of x, y and z.*

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* Kasner: "Tautochrones," *loc. cit.*, p. 480.